

IMPROVING KAUFMAN'S EXCEPTIONAL SET ESTIMATE FOR PACKING DIMENSION

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ABSTRACT. Given $0 < s < 1$, I prove that there exists a constant $\epsilon = \epsilon(s) > 0$ such that the following holds. Let $K \subset \mathbb{R}^2$ be a Borel set with $\mathcal{H}^1(K) > 0$, and let $E_s(K) \subset S^1$ be the collection of unit vectors e such that

$$\dim_p \pi_e(K) \leq s.$$

Then $\dim_H E_s(K) \leq s - \epsilon$.

1. INTRODUCTION

This paper is concerned with a classical question in fractal geometry: how do orthogonal projections affect the dimension of planar sets? For a reader not familiar with the area, I recommend the recent survey [4] of Falconer, Fraser and Jin. In this introduction, I only describe some results most relevant to the new material.

Fix $0 \leq s \leq 1$, and let $K \subset \mathbb{R}^2$ be a Borel set of Hausdorff dimension $\dim_H K \geq s$. In 1968, Kaufman [7] proved, improving an earlier result of Marstrand [9] from 1954, that

$$\dim_H \{e \in S^1 : \dim_H \pi_e(K) < s\} \leq s. \quad (1.1)$$

Here $\pi_e : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the orthogonal projection $\pi_e(x) = x \cdot e$. Under the assumption $\dim_H K \geq s$, Kaufman's bound (1.1) is sharp: in 1975, Kaufman and Mattila [8] constructed an explicit compact set $K \subset \mathbb{R}^2$ with $\dim_H K = s$ such that

$$\dim_H \{e : \dim_H \pi_e(K) < s\} = s. \quad (1.2)$$

Under the assumption $\dim_H K \geq t > s$, the sharpness of (1.1) is an open problem. The following improvement is conjectured (in (1.8) of [10], for instance):

Conjecture 1.3. *Assume that $0 \leq t/2 \leq s \leq t \leq 1$ and $\dim_H K \geq t$. Then*

$$\dim_H \{e \in S^1 : \dim_H \pi_e(K) < s\} \leq 2s - t. \quad (1.4)$$

It is known that the right hand side of (1.4) cannot be further improved.

So, what are the partial results for Conjecture 1.3, and why is the problem worth studying? The case $s \approx t/2$ attracted a great deal of attention around 2003,

2010 *Mathematics Subject Classification.* 28A75 (Primary).

T.O. is supported by the Academy of Finland through the grant Restricted families of projections and connections to Kakeya type problems.

when Edgar and Miller [3] and Bourgain [1] independently proved the Erdős-Volkmann ring conjecture. The conjecture – now a theorem – states that \mathbb{R} does not contain Borel subrings of Hausdorff dimension $r \in (0, 1)$. To explain the connection to Conjecture 1.3, assume for a moment that there existed a Borel ring $B \subset \mathbb{R}$ with $\dim_{\text{H}} B = r$ for some $0 < r \leq 1/2$. Then $\dim_{\text{H}}(B \times B) \geq 2r$, and $B + hB \subset B$ for all $h \in B$. This implies that

$$\dim_{\text{H}}\{e \in S^1 : \dim_{\text{H}} \pi_e(B \times B) \leq r\} \geq r, \quad (1.5)$$

severely violating (1.4) for s close to r and $t = 2r$. Thus, Conjecture 1.3 is stronger than the ring conjecture. In fact, the ring conjecture is no stronger than "a slight improvement over Kaufman's bound (1.1) in the case $s = t/2$ ". To see this, note that Kaufman's bound (1.1) implies

$$\dim_{\text{H}}\{e \in S^1 : \dim_{\text{H}} \pi_e(B \times B) < r\} \leq r,$$

which barely fails to contradict (1.5). So, a result of the form

$$\dim_{\text{H}}\{e : \dim_{\text{H}} \pi_e(K) \leq r\} \leq r - \epsilon \quad (1.6)$$

for $0 < r \leq (\dim_{\text{H}} K)/2$ would already be strong enough to settle the ring conjecture, and this is significantly weaker than Conjecture 1.3. Bourgain's approach to the ring problem gives (1.6), and in fact something quite a bit better, which sits between (1.6) and the conjectured bound (1.4): Theorem 3 in [2] implies that

$$\dim_{\text{H}}\{e : \dim_{\text{H}} \pi_e(K) \leq s\} \searrow 0 \quad (1.7)$$

as $s \searrow (\dim_{\text{H}} K)/2$. The biggest caveat is that (1.7) says nothing about values of s far from $(\dim_{\text{H}} K)/2$. For instance, assuming that $\dim_{\text{H}} K = 1$, the best bound for $\dim_{\text{H}}\{e : \dim_{\text{H}} \pi_e(K) \leq 3/4\}$ remains the one given by Kaufman's bound (1.1), namely $\dim_{\text{H}}\{e : \dim_{\text{H}} \pi_e(K) \leq 3/4\} \leq 3/4$.

The ring conjecture was not the only motivation for Bourgain's work [2] in 2003. Two years earlier, Katz and Tao [6] had proved that the case $t = 1/2$ of the ring conjecture is "roughly" equivalent (more precisely: equivalent at the level of certain "discretised" versions) to obtaining small improvements in Falconer's distance set problem and the $(1/2)$ -Furstenberg set problem. I recall the latter question

Question 1 (Furstenberg set problem). *Assume that a set $K \subset \mathbb{R}^2$ has the property that for every $e \in S^1$, there exists a line of the form $L_{a,e} := a + \text{span}(e)$, $a \in \mathbb{R}^2$, such that $\dim_{\text{H}}(K \cap L_{a,e}) \geq s$. Such a set K is called an s -Furstenberg set. How small can the dimension of an s -Furstenberg set be?*

Until Bourgain's work in 2003, the best result on Question 1 was due to Wolff [14], who proved that $\dim_{\text{H}} K \geq \max\{s + 1/2, 2s\}$ for every s -Furstenberg set K . In the case $s = 1/2$, Bourgain could improve Wolff's result by a small absolute constant $c > 0$, namely showing that $\dim_{\text{H}} K \geq 1 + c$. To sum up, a slight improvement of the type (1.6) for Kaufman's bound (1.1) in the case $\dim_{\text{H}} K = 1$ and $s = 1/2$ is stronger than the case $t = 1/2$ of the ring conjecture, which is,

further, "roughly" equivalent to proving an improvement for the dimension of $(1/2)$ -Furstenberg sets. This is not a rigorous argument, but it is a fair guideline.

How about s -Furstenberg sets for $s \in (1/2, 1)$? For s "very close" to $1/2$, Bourgain's approach still gives an improvement over the Wolff bound, just as (1.7) gives an improvement to Kaufman's bound for s "very close" to $1/2$. But for $s = 3/4$, say, the best dimension bound for s -Furstenberg sets remains Wolff's estimate $\min\{1/2 + s, 2s\} = 2s$. And, heuristically, improving Kaufman's bound for $\dim_H K = 1$, and any $s \in (1/2, 1)$, is "close" to improving Wolff's $2s$ -bound for the **same** value of s . Unfortunately, this is only a guideline, not an established fact; as far as I know, the only rigorous argument in this vein is contained in D. Oberlin's paper [11]. There, an improvement to Kaufman's bound for projections is shown to imply an improvement over Wolff's bound for certain "toy" Furstenberg sets, which arise from a special, if natural, construction. So, even if the the Kaufman and Furstenberg problems are perhaps not equivalent for every $s \in (1/2, 1)$, it is not outrageous exaggeration to claim that the former acts as a toy question towards the latter.

The aim of this paper is to study Kaufman's bound for $\dim_H K = 1$, and for any $1/2 < s < 1$ (the cases $s = 1/2$ and $0 \leq s < 1/2$ were solved by Bourgain [2] and Oberlin [10], respectively). Here is the main result:

Theorem 1.8. *Given $1/2 < s < 1$, there exists a constant $\epsilon = \epsilon(s) > 0$ such that the following holds. If $K \subset \mathbb{R}^2$ be a Borel set with $\mathcal{H}^1(K) > 0$, then*

$$\dim_H(\{e \in S^1 : \dim_p \pi_e(K) \leq s\}) \leq s - \epsilon.$$

Here \dim_p stands for packing dimension.

Theorem 1.8 does not improve over (1.1), since it only gives an upper bound for the dimension of $\{e : \dim_p \pi_e(K) \leq s\}$ (a subset of $\{e : \dim_H \pi_e(K) \leq s\}$). However, to the best of my knowledge, the bound " s " given by (1.1) was, up to now, the best available even for $\dim_H\{e : \dim_p \pi_e(K) \leq s\}$. The assumption $\mathcal{H}^1(K) > 0$ is a matter of convenience and could easily be relaxed to $\dim_H K = 1$. As far as I can tell, the appearance of \dim_p is quite crucial for the proof strategy, and dealing with \dim_H requires a new idea. On the other hand, there is some hope that the proof strategy behind Theorem 1.8 could give an ϵ -improvement over Wolff's bound for the upper box dimension of Furstenberg s -sets, $1/2 < s < 1$. This requires further investigation.

1.1. Outline of the proof. In short, the proof of Theorem 1.8 consists of two steps. One is to consider special sets K , which are roughly of the form $K = A \times B$, where A is s -dimensional and B is $(1 - s)$ -dimensional. For such sets, one can prove Theorem 1.8 by a direct argument, which uses tools from additive combinatorics (see Section 3 below). The second step is to reduce the proof for general sets to the special case. This involves first pigeonholing a suitable scale $\delta > 0$ to work on. Then, one makes a counter assumption (namely: K has an almost

s -dimensional set of s -dimensional projections) both at scales $\delta^{1/2}$ and δ . This evidently relies on having information about $\dim_p \pi_e(K)$ and not just $\dim_H \pi_e(K)$. If the counter assumption is strong enough, one can find the following structure inside K : there is a $(\delta^{1/2} \times 1)$ -tube T such that if one blows up $T \cap K$ into the unit square, then the resulting set \tilde{K} "behaves like a $s \times (1 - s)$ -dimensional product set with an almost s -dimensional set of s -dimensional projections". In effect, this means that the existence of \tilde{K} contradicts the result obtained in the first part of the proof. Hence, one finally obtains a contradiction.

2. PRELIMINARIES AND KAUFMAN'S BOUND

The purpose of this section is to record some preliminaries, notation and auxiliary results, and give a quick proof of the well-known and easy estimate $\dim_H\{e : \dim_p \pi_e(K) \leq s\} \leq s$ (that is, Theorem 2.1 without the ϵ -improvement).

First, I observe that in place of Theorem 1.8, it suffices to prove its analogue for *upper box dimension*:

Theorem 2.1. *Given $1/2 < s < 1$, there exists a constant $\epsilon = \epsilon(s) > 0$ such that the following holds. If $K \subset \mathbb{R}^2$ be a compact set with $\mathcal{H}^1(K) > 0$, then*

$$\dim_H(\{e \in S^1 : \overline{\dim}_B \pi_e(K) \leq s\}) \leq s - \epsilon.$$

Here $\overline{\dim}_B$ stands for the upper box (or Minkowski) dimension, which, for bounded sets $A \subset \mathbb{R}^d$, is defined by

$$\overline{\dim}_B A := \limsup_{\delta \rightarrow 0} \frac{\log N(A, \delta)}{-\log \delta}.$$

The quantity $N(A, \delta)$ is the least number of balls of radius δ required to cover A . The fact that Theorem 2.1 implies Theorem 1.8 follows immediately from Lemma 4.5 in [12].

The proof of Theorem 2.1 proceeds by counter assumption and contradiction. Namely, I will assume that $\mathcal{H}^{s-\epsilon_0/2}(\{e : \overline{\dim}_B \pi_e(K) \leq s\}) > 0$ for some (very small) $\epsilon_0 > 0$. In particular, it follows that

$$\{e \in S^1 : N(\pi_e(K), \delta) \leq \delta^{-s-\epsilon_0/2} \text{ for all } 0 < \delta \leq \delta_0\}$$

has positive $(s - \epsilon_0/2)$ -dimensional measure for small enough $\delta_0 > 0$. Replacing s by $s - \epsilon_0/2$ for notational convenience, I will assume that $\mathcal{H}^s(E) > 0$, where

$$E := \{e \in S^1 : N(\pi_e(K), \delta) \leq \delta^{-s-\epsilon_0} \text{ for all } 0 < \delta \leq \delta_0\}. \quad (2.2)$$

Throughout the paper, I will use four types of "less than" inequality signs: \leq , \lesssim , \lesssim_{\log} and \lesssim . The first is most likely familiar to the reader, while $A \lesssim B$ means that there exists a constant $C \geq 1$ such that $A \leq CB$. If the dependence of C on some parameter p should be emphasised, this will be denoted by $A \lesssim_p B$. The inequality sign $A \lesssim_{\log} B$ means that

$$A \lesssim \log^C(1/\delta)B,$$

where $C \geq 1$ is some constant (always quite small, $C \leq 10$), and $\delta > 0$ is a *scale*, whose meaning will be clear later. Finally, the notation $A \lesssim B$ means that

$$A \leq C_{\epsilon_0} \delta^{C_{\epsilon_0}} B.$$

Here ϵ_0 is the "counter assumption parameter" from (2.2), $C_{\epsilon_0} \geq 1$ is a constant depending only on ϵ_0 and "harmless parameters", and $C \geq 1$ is a constant depending only on "harmless parameters". These "harmless parameters" consist of quantities, which are regarded as "fixed" throughout the proof; a typical example is the number s . The notations $A \geq / \gtrsim / \gtrsim_{\log} / \gtrsim B$ mean that $B \leq / \lesssim / \lesssim_{\log} / \lesssim A$, and the notations $A = / \sim / \sim_{\log} / \approx B$ stand for two-sided inequalities.

If a little imprecision is allowed for a moment, the entire proof of Theorem 2.1 will consist of a finite chain of inequalities of the form $A_1 \lesssim A_2 \lesssim \dots \lesssim A_m$, and finally the observation that $A_1 \gtrsim \delta^{-\epsilon_1} A_m$ for some absolute constant $\epsilon_1 > 0$. Thus, if $\delta, \epsilon_0 > 0$ are small enough, a contradiction is reached.

The next definition contains a δ -discretised analogue of "positive t -dimensional measure":

Definition 2.3 ((δ, t) -sets). Fix $\delta, t > 0$. A finite δ -separated set $P \subset \mathbb{R}^d$ is called a (δ, t) -set, if

$$|P \cap B(x, r)| \lesssim_{\log} \left(\frac{r}{\delta}\right)^t \quad (2.4)$$

for all $x \in \mathbb{R}^d$ and $\delta \leq r \leq 1$. Here and below, $|\cdot|$ stands for cardinality. The set P is called a *generalised* (δ, t) -set, if it satisfies the following relaxed version of (2.4):

$$|P \cap B(x, r)| \lesssim \left(\frac{r}{\delta}\right)^t$$

for all $x \in \mathbb{R}^d$ and $\delta \leq r \leq 1$.

The definition of *generalised* (δ, t) -set is slightly vague, and the meaning will be best clarified in actual use below. In the proofs, a typical application is the following: a certain δ -separated set P is found, and one observes that the bound $|P \cap B(x, r)| \leq C_{\epsilon_0} \delta^{-C_{\epsilon_0}} (r/\delta)^t$ holds for all $r \geq \delta$, and some constants $C, C_{\epsilon_0} \geq 1$. Then, Definition 2.4 allows me to call P a (generalised) (δ, t) -set without cumbersome book-keeping of the constants C, C_{ϵ_0} .

The rationale behind the definition of (δ, s) -sets is the fact that large (δ, s) -sets can be found, for any $\delta > 0$, inside a set with positive s -dimensional Hausdorff content. The following proposition is Proposition A.1 in [5] (the result in [5] is stated in \mathbb{R}^3 , but the verbatim same proof works in every dimension):

Proposition 2.5. Let $\delta > 0$, and let $B \subset \mathbb{R}^2$ be a set with $\mathcal{H}_{\infty}^s(B) =: \kappa > 0$. Then, there exists a (δ, s) -set $P \subset B$ with cardinality $|P| \gtrsim \kappa \cdot \delta^{-s}$. In fact, the (δ, s) -set property (2.4) even holds with " \lesssim " instead of " \lesssim_{\log} " for P .

Now, as a warm-up for things to come, but also for real use, I present a quick proof of the easy bound $\dim_{\mathbb{H}}\{e : \dim_{\mathbb{P}} \pi_e(K)\} \leq s$. As with Theorem 1.8, it suffices to prove that $\dim_{\mathbb{H}}\{e : \overline{\dim}_{\mathbb{B}} \pi_e(K) \leq s\}$, and this follows almost immediately from the next proposition:

Proposition 2.6. Fix $\delta > 0$. Let $0 < s < 1$ and let $K \subset B(0, 1) \subset \mathbb{R}^2$ be a set with $\mathcal{H}_\infty^1(K) \sim 1$. Assume that $E \subset S^1$ is a δ -separated set with $|E| \gtrsim \delta^{-s}$. Then, there exists a vector $e \in E$ with $N(\pi_e(K), \delta) \gtrsim_{\log} \delta^{-s}$.

Proof. By Proposition 2.5, there exists a $(\delta, 1)$ -set $P \subset K$ with $|P| \sim \delta^{-1}$. It suffices to find $e \in E$ such that $N(\pi_e(P), \delta) \gtrsim_{\log} \delta^{-s}$. Let $E_0 \subset E$ be the set of vectors $e \in E$ such that the claim fails: more precisely, $N(\pi_e(P), \delta) \leq M$ for $M = c\delta^{-t}/\log^C(1/\delta)$, where $c, C > 0$ are suitable constants. It suffices to show that $|E_0| < |E|$, if $c > 0$ is small enough. Fix $e \in E_0$. Then, it is easy to check using Cauchy-Schwarz (or see the proof of Proposition 4.10 in [12]) that there exist $\gtrsim |P|^2/M \gtrsim \delta^{-2}/M$ pairs $(p_1, p_2) \in P \times P$ such that

$$|\pi_e(p_1) - \pi_e(p_2)| \leq \delta.$$

Since the lower bound $|P|^2/M$ is far greater than $|P|$ for small enough $\delta > 0$, one in fact has the same lower bound for pairs (p_1, p_2) satisfying additionally $p_1 \neq p_2$. Consequently,

$$\sum_{e \in E_0} |\{(p_1, p_2) \in P \times P : p_1 \neq p_2 \text{ and } |\pi_e(p_1) - \pi_e(p_2)| \leq \delta\}| \gtrsim \frac{|E_0|}{\delta^2 M}.$$

On the other hand, using the geometric fact that $\{e \in S^1 : |\pi_e(p_1) - \pi_e(p_2)| \leq \delta\}$ is the union of two arcs of length $\lesssim \delta/|p_1 - p_2|$, one has

$$\begin{aligned} & \sum_{e \in E} |\{(p_1, p_2) \in P \times P : p_1 \neq p_2 \text{ and } |\pi_e(p_1) - \pi_e(p_2)| \leq \delta\}| \\ &= \sum_{p_1 \neq p_2} |\{e \in E : |\pi_e(p_1) - \pi_e(p_2)| \leq \delta\}| \\ &\lesssim \sum_{p_1 \neq p_2} \frac{1}{|p_1 - p_2|} \lesssim \sum_{p_1} \sum_{\delta \leq 2^j \leq 1} 2^{-j} |P \cap B(p_1, 2^j)| \\ &\lesssim_{\log} \sum_{p_1} \sum_{\delta \leq 2^j \leq 1} 2^{-j} \cdot \frac{2^j}{\delta} \lesssim_{\log} \delta^{-2}. \end{aligned}$$

Comparing the lower and upper bounds leads to

$$|E_0| \lesssim_{\log} M = \frac{c\delta^{-t}}{\log^C(1/\delta)}.$$

For $c > 0$ sufficiently small and $C \geq 1$ sufficiently large, this gives $|E_0| < |E|$, and the proof is complete. \square

Corollary 2.7. If $\mathcal{H}^1(K) > 0$, then $\dim_{\mathbb{H}}\{e \in S^1 : \overline{\dim}_{\mathbb{B}}\pi_e(K) \leq s\} \leq s$.

Proof. If the statement fails, then $\mathcal{H}^{s+2\epsilon}(\{e : \overline{\dim}_{\mathbb{B}}\pi_e(K) \leq s\}) > 0$ for some $\epsilon > 0$. By definition of $E_s(K)$, this implies that the set

$$E_i := \{e : N(\pi_e(K), \delta) \leq \delta^{-s-\epsilon} \text{ for all } \delta \leq 1/i\}$$

has positive $(s + 2\epsilon)$ -dimensional measure for some $i \in \mathbb{N}$. In particular, E_i contains a δ -separated set of cardinality $\gtrsim \delta^{-s-2\epsilon}$ for all $\delta \leq 1/i$. For small enough $\delta > 0$, this violates Proposition 2.6. \square

Remark 2.8. Proposition 2.6 is also crucial for the proof of the main theorem. Recall the set E in the main counter assumption (2.2). Then

$$N(E, \delta) \leq \delta^{-s-2\epsilon_0} \quad (2.9)$$

for small enough $\delta \leq \delta_0$. Indeed, in the opposite case Proposition 2.6 would imply that $N(\pi_e(K), \delta) \gtrsim_{\log} \delta^{-s-2\epsilon_0}$ for some $e \in E$, violating the definition of E for small enough $\delta > 0$. For simplicity and without loss of generality, I will assume that (2.9) holds for all $0 < \delta \leq \delta_0$.

3. PRODUCT-LIKE SETS

The main result of this section is a technical statement, Proposition 3.1, about "product-like" sets, which will be useful later on in the context of general sets. A simple qualitative corollary of Proposition 3.1 would state the following. Assume that $A \subset \mathbb{R}$ is s -dimensional and $B \subset \mathbb{R}$ is τ -dimensional, $\tau > 0$. Then, for any s -dimensional set $E \subset S^1$, there exists $e \in E$ such that

$$\overline{\dim}_B \pi_e(A \times B) \geq s + \epsilon,$$

where $\epsilon > 0$ only depends on s and τ . Here is the quantitative version:

Proposition 3.1. *Let $0 < s < 1$, $\epsilon, \tau > 0$. Let $B \subset [0, 1]$ be a (δ, τ) -set of cardinality $|B| \gtrsim \delta^{-\tau+\epsilon}$, and let $E \subset S^1$ be a (δ, s) -set of cardinality $|E| \gtrsim \delta^{-s+\epsilon}$. For each $b \in B$, assume that $A_b \subset [0, 1]$ is a (δ, s) -set of cardinality $|A_b| \gtrsim \delta^{-s+\epsilon}$, and let P be the $(\delta, s + \tau)$ -set*

$$P := \bigcup_{b \in B} A_b \times \{b\}. \quad (3.2)$$

Then, if ϵ is small enough (depending only on τ, s), then

$$N(\pi_e(P), \delta) \geq \delta^{-s-\epsilon} \quad \text{for some } e \in E, \quad (3.3)$$

for all sufficiently small $\delta > 0$ (depending only on s, τ , and the implicit constants behind the \sim notation).

Remark 3.4. In this section, Section 3, the constant ϵ_0 from the main counter assumption (2.2) does not make an appearance. So, it will cause no confusion, if the notations \lesssim , \gtrsim and \approx are temporarily re-purposed for the needs of Proposition 3.1. In particular, the failure of (3.3) will be denoted by $N(\pi_e(P), \delta) \lesssim \delta^{-s}$, as in (3.8) below. Similarly, the cardinality of B is $|B| \approx \delta^{-\tau}$ and so on.

Before starting the proof, I recall two standard results from additive combinatorics. The first is the *Balog-Szemerédi-Gowers theorem*. The statement below is taken verbatim from p. 196 in [2]. For a proof, see [13], p. 267.

Theorem 3.5 (Balog-Szemerédi-Gowers). *There exists an absolute constant $C \geq 1$ such that the following holds. Let $A, B \subset \mathbb{R}$ be finite sets, and assume that $G \subset A \times B$ is a set of pairs such that*

$$|G| \geq \frac{|A||B|}{K} \quad \text{and} \quad |\{x + y : (x, y) \in G\}| \leq K|A|^{1/2}|B|^{1/2}$$

for some $K > 1$. Then, there exist $A' \subset A$ and $B' \subset B$ satisfying

- $|A'| \geq K^{-C}|A|$, $|B'| \geq K^{-C}|B|$,
- $|A' + B'| \leq K^C|A|^{1/2}|B|^{1/2}$, and
- $|G \cap (A' \times B')| \geq K^{-C}|A||B|$.

The second auxiliary result is the Plünnecke-Ruzsa inequality, whose proof can also be found in [13]:

Theorem 3.6 (Plünnecke-Ruzsa). *Assume that $A, B \subset \mathbb{R}$ are finite sets such that*

$$|A + B| \leq C|A|$$

for some integer $C \geq 1$. Then

$$|B^m \pm B^n| \leq C^{m+n}|A|$$

for all $m, n \in \mathbb{N}$.

Remark 3.7. Theorem 3.6 will be applied in the following form: if $A, B \subset \mathbb{R}$ are δ -separated sets with $|A| \approx |B|$ and

$$N(A + B, \delta) \lesssim |A|,$$

then $N(B + B, \delta) \lesssim |A|$. This statement follows easily from Theorem 3.6 by considering the sets $[A]_\delta = \{[a]_\delta : a \in A\} \subset \delta\mathbb{Z}$ and $[B]_\delta := \{[b]_\delta : b \in B\} \subset \delta\mathbb{Z}$, where $[x]_\delta \in \delta\mathbb{Z}$ stands for the largest number $\delta n \in \delta\mathbb{Z}$ satisfying $\delta n \leq x$. Then the hypothesis $N(A + B, \delta) \lesssim |A|$ implies that $|[A]_\delta + [B]_\delta| \lesssim |[A]_\delta|$, so Theorem 3.6 can be applied.

Proof of Proposition 3.1. For later technical convenience, I will already make the assumption that all the vectors in E are "roughly horizontal", which precisely means the following: $e^1 \sim 1$ for all $(e^1, e^2) \in E$, and a δ -tube perpendicular to any one of the vectors $e \in E$ contains at most one point of the form $(a, b) \in A_b \times \{b\}$ for any fixed $b \in B$. This can be arranged by replacing E and the sets A_b by suitable subsets $A'_b \subset A_b$ and $E' \subset E$ with $|A'_b| \sim |A|$ and $|E'| \sim |E|$.

Another convenient extra hypothesis is that $A_b \subset \delta\mathbb{Z}$ for all $b \in B$. This can be achieved by replacing the sets A_b by the sets $[A_b]_\delta := \{[a]_\delta : a \in A_b\}$. Neither the hypotheses nor the conclusion of the theorem are relevantly affected by the passage from A_b to $[A_b]_\delta$, since the sets A_b were assumed to be δ -separated to begin with.

The proof can now start in earnest. I make the counter assumption that

$$N(\pi_e(P), \delta) \lesssim \delta^{-s}, \quad e \in E, \tag{3.8}$$

and gradually work towards a contradiction. Fix a vector $e_0 = (e_0^1, e_0^2) \in E$, and write $A := \pi_{e_0}(P)$, so that

$$N(A, \delta) \approx \delta^{-s} \quad (3.9)$$

by (3.8). I will first argue that one may assume $e_0 = (1, 0)$ without loss of generality. Note that

$$e_0^1 A_b + e_0^2 b \subset \pi_{e_0}(P) \subset A$$

for every $b \in B$. Now, if A_b is replaced by $A'_b := e_0^1 A_b + e_0^2 b$, and P' is built from these A'_b as in (3.2), then it is clear that P' is of the form discussed in the statement of the theorem, and

$$\pi_{(1,0)}(P') \subset A,$$

and (3.8) holds for P' and the vectors $E' := \{(e^1/e_0^1, e^2 - e^1[e_0^2/e_0^1]) : (e^1, e^2) \in E\}$. Since E' is obtained from E by an affine transformation of determinant $1/e_0^1 \sim 1$, one sees that E' is a (δ, s) -set of cardinality $\approx \delta^{-s}$. Thus, one can first prove the theorem for P' instead of P , and finally do the affine transformation in the other direction to get the result for P . So, assume without loss of generality that $\pi_{(1,0)}(P) \subset A$, which implies that

$$A_b \subset A, \quad b \in B. \quad (3.10)$$

Finally, since one was also allowed to assume $A_b \subset \delta\mathbb{Z}$, it follows from (3.9) that

$$A \subset \delta\mathbb{Z} \quad \text{and} \quad |A| \approx \delta^{-s}. \quad (3.11)$$

For each $e \in E$, cover P by $\lesssim \delta^{-s}$ tubes of dimensions $\delta \times 10$, perpendicular to e . Denote these tubes by \mathcal{T}_e , and write

$$\mathcal{T} := \bigcup_{e \in E} \mathcal{T}_e,$$

so that $|\mathcal{T}| \lesssim \delta^{-2s}$. Then, for $e \in E$ and distinct $p, q \in P$, write $p \sim_e q$, if there exists $T \in \mathcal{T}_e$ such that $p, q \in T$. Further, define $p \sim q$, if $p \sim_e q$ for some $e \in E$ (that is, $p, q \in T$ for some $T \in \mathcal{T}$). The first task is to find a lower bound for the number of pairs

$$Q := \{(p, q) \in P \times P : p \sim q\}.$$

The desired estimate is $|Q| \gtrsim \delta^{-2s-2\tau} \approx |P|^2$. To this end, note that for fixed $e \in E$, it is easy to check (using Cauchy-Schwarz) that

$$|\{(p, q) \in P \times P : p \sim_e q\}| \gtrsim \delta^{-s-2\tau},$$

so that

$$\sum_{e \in E} |\{(p, q) \in P \times P : p \sim_e q\}| \gtrsim \delta^{-2s-2\tau} \approx |P|^2. \quad (3.12)$$

This almost looks like the desired estimate, but the sets in the summation need not be disjoint for distinct $e \in E$. However, using the (δ, s) -set property of E (and

the geometry of $\{e \in S^1 : p \sim_e q\}$, the left hand side of (3.12) can be estimated from above as follows:

$$\begin{aligned} \text{L.H.S of (3.12)} &= \sum_{(p,q) \in Q} |\{e \in E : p \sim_e q\}| \lesssim \sum_{(p,q) \in Q} \frac{1}{|p - q|^s} \\ &\leq |Q|^{1/q} \left(\sum_{p \neq q} \frac{1}{|p - q|^{s+\tau}} \right)^{1/p} \lesssim_{\log} |Q|^{1/q} |P|^{2/p}, \end{aligned}$$

where $p > 1$ is chosen so that $ps = s + \tau$, and the last inequality follows from the fact that P is a $(\delta, s + \tau)$ -set of cardinality $\approx \delta^{-s-\tau}$. It follows from this and (3.12) that

$$|Q| \gtrsim |P|^2, \quad (3.13)$$

as claimed. Further, note that

$$\sum_{b_1 \neq b_2} |\{(p, q) \in A^{b_1} \times A^{b_2} : p \sim q\}| = |Q| \gtrsim |P|^2, \quad (3.14)$$

where $A^{b_i} := A_{b_i} \times \{b_i\} \subset P$. This follows from (3.13) and the fact that the tubes in \mathcal{T} are fairly vertical (so that there are no relations $p \sim q$ with $p, q \in A^b$).

Fixing b_1, b_2 , let $\mathcal{T}_{b_1, b_2} \subset \mathcal{T}$ be a collection of tubes such that every pair $(p, q) \in A^{b_1} \times A^{b_2}$ with $p \sim q$ is contained in a tube in \mathcal{T}_{b_1, b_2} . Such tubes exist by definition of the relation " \sim ", but they need not be unique: pick exactly one tube $T_{(p, q)}$ for every pair $(p, q) \in A^{b_1} \times A^{b_2}$. Then

$$|\mathcal{T}_{b_1, b_2}| \geq |\{(p, q) \in A^{b_1} \times A^{b_2} : p \sim q\}|,$$

because the mapping $(p, q) \mapsto T_{(p, q)}$ is injective by the assumption that the vectors e are "roughly horizontal" (see the first paragraph of the proof for a precise statement). Consequently, by (4.9),

$$\sum_{b_1, b_2} |\mathcal{T}_{b_1, b_2}| \gtrsim |P|^2. \quad (3.15)$$

(Here $\mathcal{T}_{b, b} := \emptyset$ for $b \in B$.) Since $\mathcal{T}_{b_1, b_2} \subset \mathcal{T}$, and $|\mathcal{T}| \lesssim \delta^{-2s}$, one sees from (3.15) that $|\mathcal{T}_{b_1, b_2}| \approx \delta^{-2s}$ for "most" pairs $(b_1, b_2) \in B^2$. In fact, something slightly better

is needed, and follows from the next Cauchy-Schwarz estimate, and $|\mathcal{T}| \lesssim \delta^{-2s}$:

$$\begin{aligned}
\sum_{b_1, b_2, b_3} |\mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}| &= \sum_{T \in \mathcal{T}} \sum_{b_2} \sum_{b_1, b_3} \chi_{\mathcal{T}_{b_1, b_2}}(T) \chi_{\mathcal{T}_{b_2, b_3}}(T) \\
&= \sum_{T \in \mathcal{T}} \sum_{b_2} \left(\sum_b \chi_{\mathcal{T}_{b, b_2}}(T) \right)^2 \\
&\geq \frac{1}{|\mathcal{T}| |B|} \left(\sum_{T \in \mathcal{T}} \sum_{b, b_2} \chi_{\mathcal{T}_{b, b_2}}(T) \right)^2 \\
&\gtrsim \frac{|P|^4}{|\mathcal{T}| |B|} \gtrsim \delta^{-2s} |B|^3.
\end{aligned}$$

Since $|\mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}| \lesssim \delta^{-2s}$ for any triple (b_1, b_2, b_3) , it follows that there exist $\approx |B|^3$ triples (b_1, b_2, b_3) with the property that $|\mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}| \approx \delta^{-2s}$. As will be made precise in a moment, the condition $|\mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}| \approx \delta^{-2s}$ roughly means that there are $\approx \delta^{-2s}$ points in $A_{b_1} \times A_{b_2}$ such that the projection of these points is small in a certain direction, determined by b_1, b_2, b_3 .

Consider a triple $(b_1, b_2, b_3) \in B^3$ with $|\mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}| \approx \delta^{-2s}$. Fix a tube $T \in \mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}$. Since $T \in \mathcal{T}_{b_1, b_2}$, one has $T = T_{(p_1, q)}$ for some (unique) pair of points

$$p_1 = (a_1, b_1) \in A_{b_1} \times \{b_1\} \quad \text{and} \quad q = (a_2, b_2) \in A_{b_2} \times \{b_2\}.$$

Similarly, because $T \in \mathcal{T}_{b_2, b_3}$, there exists yet another (unique) point

$$p_3 = (a_3, b_3) \in A_{b_3} \times \{b_3\}$$

such that $T = T_{(q, p_3)}$. In particular, gathering all the pairs $(a_1, a_3) \in A_{b_1} \times A_{b_3}$ obtained this way, one sees that the tubes $T \in \mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}$ give rise to a subset

$$G'_{b_1, b_2, b_3} \subset A_{b_1} \times A_{b_3} \stackrel{(3.10)}{\subset} A \times A$$

of cardinality

$$|G'_{b_1, b_2, b_3}| = |\mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}| \approx \delta^{-2s} \approx |A|^2.$$

From now on, restrict attention to triples $(b_1, b_2, b_3) \in B^3$ such that

$$\min_{i \neq j} |b_i - b_j| \approx 1. \tag{3.16}$$

Since the triples **failing** this condition have cardinality far less than $|B|^3$ (using the (δ, τ) -set hypothesis of B , and the assumption $|B| \approx \delta^{-\tau}$), one sees that $|\mathcal{T}_{b_1, b_2} \cap \mathcal{T}_{b_2, b_3}| \approx \delta^{-2s}$ holds for $\approx |B|^3$ triples satisfying (3.16). Fix one such triple, assume that $b_1 < b_3$, and consider a pair $(a_1, a_3) \in G'_{b_1, b_2, b_3}$. Recall how such points arise, and the notation for p_1, q, p_3 . Let

$$L = \left\{ x = \frac{a_3 - a_1}{b_3 - b_1} y + \frac{a_1 b_3 - a_3 b_1}{b_3 - b_1} : y \in \mathbb{R} \right\}$$

be the line spanned by p_1 and p_3 ; then, since p_1, q, p_3 all lie in the common δ -tube T , the line L passes at distance $\lesssim \delta$ from $q = (a_2, b_2) \in A_{b_2} \times \{b_2\}$, which is equivalent to

$$\left| \frac{a_3(b_2 - b_1) + a_1(b_3 - b_2)}{b_3 - b_1} - a_2 \right| \lesssim \delta.$$

Recalling (3.16), this further implies that

$$\left| \left(a_1 + \frac{b_2 - b_1}{b_3 - b_2} a_3 \right) - \frac{b_3 - b_1}{b_3 - b_2} a_2 \right| \lesssim \delta.$$

Consequently, if π_{b_1, b_2, b_3} stands for the projection-like mapping

$$\pi_{b_1, b_2, b_3}(x, y) = x + \frac{b_2 - b_1}{b_3 - b_2} y, \quad (3.17)$$

then

$$\text{dist} \left(\pi_{b_1, b_2, b_3}(G'_{b_1, b_2, b_3}), \frac{b_3 - b_1}{b_3 - b_2} A_{b_2} \right) \lesssim \delta.$$

Observing $N([(b_3 - b_1)/(b_3 - b_2)]A_{b_2}, \delta) \lesssim \delta^{-s}$, it follows that

$$N(\pi_{b_1, b_2, b_3}(G'_{b_1, b_2, b_3}), \delta) \lesssim \delta^{-s} \approx |A|. \quad (3.18)$$

In fact, this holds for any triple $(b_1, b_2, b_3) \in B^3$ satisfying (3.16) by definition of G'_{b_1, b_2, b_3} , but the information is most useful, if $|G'_{b_1, b_2, b_3}| \approx |A|^2$. Write

$$F_{b_1, b_2, b_3} := \left\{ \left(a_1, \left[\frac{b_2 - b_1}{b_3 - b_2} a_2 \right]_\delta \right) : (a_1, a_2) \in G'_{b_1, b_2, b_3} \right\} \subset A \times \left[\frac{b_2 - b_1}{b_3 - b_2} A \right]_\delta.$$

It follows easily from (3.18) (and recalling $A \subset \delta\mathbb{Z}$) that

$$|\{f_1 + f_2 : (f_1, f_2) \in F_{b_1, b_2, b_3}\}| \lesssim |A|.$$

Moreover, since $|(b_2 - b_1)/(b_3 - b_2)| \approx 1$ for every triple (b_1, b_2, b_3) satisfying (3.16), it follows that $|F_{b_1, b_2, b_3}| \approx |A|^2$ whenever (3.16) holds and $|G'_{b_1, b_2, b_3}| \approx |A|^2$. For such a *good* triple (b_1, b_2, b_3) , the Balog-Szemerédi-Gowers theorem, Theorem 3.5, implies that there exist subsets

$$D_{b_1, b_2, b_3}^1 \subset A \quad \text{and} \quad \tilde{D}_{b_1, b_2, b_3}^2 \subset \left[\frac{b_2 - b_1}{b_3 - b_2} A \right]_\delta$$

such that $|D_{b_1, b_2, b_3}^1|, |\tilde{D}_{b_1, b_2, b_3}^2| \approx |A|$,

$$|(D_{b_1, b_2, b_3}^1 \times \tilde{D}_{b_1, b_2, b_3}^2) \cap F_{b_1, b_2, b_3}| \approx |A|^2 \quad (3.19)$$

and

$$|D_{b_1, b_2, b_3}^1 + \tilde{D}_{b_1, b_2, b_3}^2| \lesssim |A|. \quad (3.20)$$

Let

$$D_{b_1, b_2, b_3}^2 := \left\{ a \in A : \left[\frac{b_2 - b_1}{b_3 - b_2} a \right]_\delta \in \tilde{D}_{b_1, b_2, b_3}^2 \right\}.$$

It then follows from the definition of F_{b_1, b_2, b_3} and (3.19) that

$$|G_{b_1, b_2, b_3}| := |(D_{b_1, b_2, b_3}^1 \times D_{b_1, b_2, b_3}^2) \cap G'_{b_1, b_2, b_3}| \approx |A|^2. \quad (3.21)$$

for a good triple (b_1, b_2, b_3) . Moreover, (3.20) easily implies that

$$N_1 := N \left(D_{b_1, b_2, b_3}^1 + \frac{b_2 - b_1}{b_3 - b_2} D_{b_1, b_2, b_3}^2, \delta \right) \lesssim |A|. \quad (3.22)$$

Finally, combining (3.22) with the Plünnecke-Ruzsa inequality, Theorem 3.6, gives

$$N_2 := (D_{b_1, b_2, b_3}^2 + D_{b_1, b_2, b_3}^2, \delta) \lesssim |A| \quad (3.23)$$

for any good triple (b_1, b_2, b_3) . Since there are $\approx |B|^3$ good triples (b_1, b_2, b_3) , one can find b_2, b_3 such that (3.21)–(3.23) hold for $\approx |B|$ choices of b_1 . Fix such $b_2, b_3 \in B$. Then, a simple Cauchy-Schwarz argument shows that $|G_{b_1, b_2, b_3} \cap G_{b'_1, b_2, b_3}| \approx |A|^2$ for $\approx |B|^2$ pairs (b_1, b'_1) , so that one can finally also fix $b_1 \in B$ such that

$$|G_b| := |G_{b_1, b_2, b_3} \cap G_{b, b_2, b_3}| \approx |A|^2 \quad (3.24)$$

for $\approx |B|$ choices of $b \in B$. For this specific (good triple) (b_1, b_2, b_3) , I denote the set of $b \in B$ such that (3.24) holds by B_0 . With (3.22) in mind, write

$$c_b := \frac{b_2 - b}{b_3 - b_2}, \quad b \in B_0,$$

and abbreviate $c := c_{b_1}$ (note that $|c|, |c_b| \approx 1$ for all $b \in B_0$ by (3.16)). Also, write

$$D^1 := D_{b_1, b_2, b_3}^1(\delta) \quad \text{and} \quad D^2 := D_{b_1, b_2, b_3}^2(\delta),$$

where $R(\delta)$ stands for the δ -neighbourhood of $R \subset \mathbb{R}^d$. To complete the proof, I repeat an argument of Bourgain (see p. 219 in [2]). Assume for a moment that $x \in cD^2 \times D^2 \subset \mathbb{R}^2$ and $b \in B_0$. Then $\chi_{-G_b(\delta)-y}(x) = 1$, whenever

$$y \in -G_b(\delta) - x \subset -(D^1 \times D^2) - (cD^2 \times D^2) = -(D^1 + cD^2) \times -(D^2 + D^2),$$

(the first inclusion uses (3.21) and (3.24)) and the Lebesgue measure of such choices y is evidently $\mathcal{L}^2(G_b(\delta))$. This gives the inequality

$$\chi_{cD^2 \times D^2} \leq \frac{1}{\mathcal{L}^2(G_b(\delta))} \int_{-(D^1 + cD^2) \times -(D^2 + D^2)} \chi_{-G_b - y} dy$$

which easily implies

$$\chi_{cD^2 + c_b D^2} \leq \frac{1}{\mathcal{L}^2(G_b(\delta))} \int_{-(D^1 + cD^2) \times -(D^2 + D^2)} \chi_{\pi_{b, b_2, b_3}(-G_b) - \pi_{b, b_2, b_3}(y)} dy, \quad b \in B_0,$$

by the definition of π_{b, b_2, b_3} (see (3.17)). Finally, integrating the previous inequality and recalling (3.22), (3.23) and (3.18), one obtains

$$\mathcal{L}^1(cD^2 + c_b D^2) \lesssim \frac{(N_1 \delta)(N_2 \delta)}{\mathcal{L}^2(G_b(\delta))} \mathcal{L}^1(\pi_{b_1, b, b_2}(G_b)) \lesssim \delta^{1-s} \approx \delta |A|, \quad b \in B_0. \quad (3.25)$$

However, $cD^2 \times D^2$ is the δ -neighbourhood of a generalised $(\delta, 2s)$ -set in the plane, so Bourgain's discretized projection theorem, Theorem 5 in [2], can be applied with $\alpha := 2s < 2 =: d$ and any $\kappa > 0$. If μ_1 is the natural probability measure on the δ -neighbourhood of $\{c_b : b \in B_0\}$, then μ_1 satisfies assumption (0.14) from [2] for any $\tau_0 > 0$ (recall the definition of the numbers c_b , in particular $|b_2 - b_3| \approx 1$,

recall that $B_0 \subset B$ has cardinality $|B_0| \approx |B|$, and B is a (δ, τ) -set). The conclusion (in (0.19) of [2]) is that some $b \in B_0$ should violate (3.25). Thus, a contradiction is reached, and the proof is complete. \square

4. GENERAL SETS

So far, the scale $\delta > 0$ has been small but otherwise arbitrary. To prove Theorem 2.1, one needs to deal with a set $K \subset B(0, 1)$ with $\mathcal{H}^1(K) > 0$. To extract useful information from the main counter assumption (2.2), namely that

$$N(\pi_e(K), \delta) \leq \delta^{-s-\epsilon_0}, \quad 0 < \delta \leq \delta_0, \quad (4.1)$$

for all $e \in E$ with $\mathcal{H}^s(E) > 0$, I will need a special scale $\delta > 0$ with the properties that K looks approximately 1-dimensional (in a rather weak sense) both at scales $\delta^{1/2}$ and δ . Such a scale can be found with a pigeonholing argument, given in the first subsection below. Then, since the counter assumption concerns all (small) scales $\delta > 0$, it applies in particular to the specific scale the pigeon helped to find.

4.0.1. *Choosing the scale δ .* By choosing a subset of K , one may assume that $0 < \mathcal{H}^1(K) < \infty$. I treat $\mathcal{H}^1(K)$ as an absolute constant, so that $\mathcal{H}^1(K) \sim 1$. Let μ be a Frostman measure supported on K , that is, $\mu(K) = 1$ and $\mu(B(x, r)) \lesssim r$ for all balls $B(x, r) \subset \mathbb{R}^2$. Next, let \mathcal{B} be an efficient δ_0 -cover for K , that is,

$$\sup\{\text{diam } B : B \in \mathcal{B}\} \leq \delta_0 \quad \text{and} \quad \sum_{B \in \mathcal{B}} \text{diam}(B) \lesssim \mathcal{H}^1(K) \sim 1. \quad (4.2)$$

For $j \in \mathbb{N}$ such that $2^{-j} \leq \delta_0$, set $\mathcal{B}_j := \{B \in \mathcal{B} : \text{diam}(B) \sim 2^{-j}\}$, and observe that

$$\sum_{2^{-j} \leq \delta_0} \sum_{B \in \mathcal{B}_j} \mu(B) \geq \mu(K) = 1.$$

In particular, there exists an index $j \in \mathbb{N}$ with $2^{-j} \leq \delta_0$ and

$$\sum_{B \in \mathcal{B}_j} \mu(B) \gtrsim \frac{1}{(j - j_0 + 1)^2}. \quad (4.3)$$

Here $j_0 \in \mathbb{N}$ satisfies $2^{-j_0} \sim \delta_0$. Now, I declare that

$$\delta := 2^{-2j},$$

so that $\delta^{1/2} = 2^{-j}$. In particular, (4.3) implies that

$$\sum_{B \in \mathcal{B}_j} \mu(B) \gtrsim_{\log} 1. \quad (4.4)$$

Observe that $|\mathcal{B}_j| \lesssim \delta^{-1/2}$ by (4.2), and on the other hand every ball $B \in \mathcal{B}_j$ satisfies $\mu(B) \lesssim \delta^{1/2}$. Thus, (4.4) implies that there are $\sim_{\log} \delta^{-1/2}$ balls in \mathcal{B}_j , denoted by \mathcal{B}_j^G , such that

$$\mu(B) \gtrsim_{\log} \delta^{1/2}, \quad B \in \mathcal{B}_j^G. \quad (4.5)$$

Discarding a few balls if necessary, one may assume that the

$$\text{dist}(B, B') \geq \delta^{-1/2}, \quad B, B' \in \mathcal{B}_j^G. \quad (4.6)$$

For each ball $B \in \mathcal{B}_j^G$, choose a $(\delta, 1)$ -set $P_B \subset B$ with $|P_B| \gtrsim_{\log} \delta^{-1/2}$. This is possible by Proposition 2.5, since (4.5) and the linear growth of μ imply that $\mathcal{H}_\infty^1(B \cap K) \gtrsim_{\log} \delta^{1/2}$.

For each ball $B \in \mathcal{B}_j^G$, pick a single point $p_B \in P_B$, and write $P_{\delta^{1/2}} := \{p_B : B \in \mathcal{B}_j^G\}$. Then $|P_{\delta^{1/2}}| \sim_{\log} \delta^{-1/2}$. Also, write

$$P := P_\delta := \bigcup_{B \in \mathcal{B}_j^G} P_B.$$

Then $|P_\delta| \sim_{\log} \delta^{-1}$, and $P_B = B \cap P$. I conclude the section by verifying that $P_{\delta^{1/2}}$ is a $(\delta^{1/2}, 1)$ -set, and P is a $(\delta, 1)$ -set. Towards the first claim, fix $x \in \mathbb{R}^2$ and $\delta^{1/2} \leq r \leq 1$. Then, writing $M := |B(x, r) \cap P_{\delta^{1/2}}|$, observe that

$$r \gtrsim \mu(B(x, 2r)) \geq \sum_{p_B \in B(x, r) \cap P_{\delta^{1/2}}} \mu(B) \gtrsim_{\log} M \delta^{1/2}$$

by (4.5). This gives $M \lesssim_{\log} r / \delta^{1/2}$, as desired. Next, consider the claim for P . For $\delta \leq r \leq \delta^{1/2}$, note that

$$|B(x, r) \cap P| = |B(x, r) \cap P_B| \lesssim_{\log} \frac{r}{\delta}$$

by (4.6) and the fact that P_B is a $(\delta, 1)$ -set. Finally, for $\delta^{1/2} \leq r \leq 1$, observe that

$$\frac{r}{\delta} \gtrsim_{\log} |P_{\delta^{1/2}} \cap B(x, 2r)| \cdot \delta^{-1/2} \gtrsim_{\log} |P \cap B(x, r)|,$$

since for every point in $p \in P \cap B(x, r)$, one has $p \in P_B$ for a certain $B \in \mathcal{B}_j^G$, and then $p_B \in B(x, 2r) \cap P_{\delta^{1/2}}$.

I recap the achievements so far. For a certain scale $\delta \leq (\delta_0)^2$, the following hold:

- $P \subset K$ is a $(\delta, 1)$ -set of cardinality $|P| \sim_{\log} \delta^{-1}$.
- P can be covered by $\sim_{\log} \delta^{-1/2}$ balls of diameter $\sim \delta^{1/2}$ in the collection \mathcal{B}_j^G , which I will henceforth denote simply by \mathcal{B} . For every $B \in \mathcal{B}$, the set P contains a special point p_B , and the set $P_{\delta^{1/2}} \subset P$ of these special points is a $(\delta^{1/2}, 1)$ -set of cardinality $|P_{\delta^{1/2}}| \sim_{\log} \delta^{-1/2}$.
- Since $P_{\delta^{1/2}} \subset P \subset K$ and $\delta^{1/2} \leq \delta_0$, the main counter assumption (4.1) implies that

$$N(\pi_e(P_{\delta^{1/2}}), \delta^{1/2}) \leq \delta^{-(s+\epsilon_0)/2} \quad (4.7)$$

and

$$N(\pi_e(P), \delta) \leq \delta^{-s-\epsilon_0} \quad (4.8)$$

for $e \in E$.

4.0.2. *The sets E and $E_{\delta^{1/2}}$.* Recall from Remark 2.8 that

$$N(E, \delta') \leq (\delta')^{-s-2\epsilon_0}, \quad \delta' \leq \delta_0.$$

This will presently be applied with $\delta' = \delta^{1/2}$, where $\delta > 0$ is the fixed scale from the discussion above. Since $\mathcal{H}^s(E) > 0$, one can find (by Proposition 2.5) a (δ, s) -subset of cardinality $\sim \delta^{-s}$. This finite subset will henceforth be denoted by E ; note that (4.7) and (4.8) remain trivially valid. Since $N(E, \delta^{1/2}) \leq \delta^{-s/2-\epsilon_0}$, and every arc of length $\delta^{1/2}$ can only contain $\lesssim \delta^{-s/2}$ points in E , it follows that there exist at least $\gtrsim \delta^{-s/2}$ arcs $J_1, \dots, J_N \subset S^1$ of length $\delta^{1/2}/10$ with

$$|E \cap J_i| \gtrsim \delta^{-s/2+\epsilon_0} \gtrsim \delta^{-s/2}.$$

For every arc J_i , pick a single point, and denote the set thus obtained by $E_{\delta^{1/2}}$. By discarding a few points, one may assume that $E_{\delta^{1/2}}$ is $\delta^{1/2}$ -separated, and $|E_{\delta^{1/2}}| \sim \delta^{-s/2}$. Moreover, $E_{\delta^{1/2}}$ is a generalised $(\delta^{1/2}, s)$ -set, since for $x \in E_{\delta^{1/2}}$ and $r \geq \delta^{1/2}$, one has

$$\left(\frac{r}{\delta}\right)^s \gtrsim |E \cap B(x, r)| \gtrsim \delta^{-s/2} |E_{\delta^{1/2}} \cap B(x, r)|.$$

4.0.3. *The distribution of P in $\delta^{1/2}$ -tubes.* Note that (4.7) holds for all $e \in E_{\delta^{1/2}}$. Thus, for every $e \in E_{\delta^{1/2}}$, the set $P_{\delta^{1/2}}$ is covered by a collection of $\leq \delta^{-s/2-\epsilon_0} \lesssim \delta^{-s/2}$ tubes \mathcal{T}_e of width $\delta^{1/2}$ and perpendicular to e . The next goal is to show that, for a typical choice of $e \in E_{\delta^{1/2}}$ and $T \in \mathcal{T}_e$, the set $T \cap P_{\delta^{1/2}}$ is essentially a $(\delta^{1/2}, 1-s)$ -set. This is a consequence of the next estimate:

$$\begin{aligned} & \frac{1}{|E_{\delta^{1/2}}|} \sum_{e \in E_{\delta^{1/2}}} \sum_{T \in \mathcal{T}_e} \sum_{\substack{p, q \in T \cap P_{\delta^{1/2}} \\ p \neq q}} \frac{1}{|p - q|^{1-s}} \\ & \leq \frac{1}{|E_{\delta^{1/2}}|} \sum_{\substack{p, q \in P_{\delta^{1/2}} \\ p \neq q}} \frac{1}{|p - q|^{1-s}} \sum_{e \in E_{\delta^{1/2}}} \chi_{\{p, q \in T \text{ for some } T \in \mathcal{T}_e\}} \\ & \lesssim \frac{1}{|E_{\delta^{1/2}}|} \sum_{\substack{p, q \in P_{\delta^{1/2}} \\ p \neq q}} \frac{1}{|p - q|} \sim_{\log} \delta^{s/2-1}. \end{aligned}$$

In passing between the second and third line, the (generalised) $(\delta^{1/2}, s)$ -set property of $E_{\delta^{1/2}}$ was used, while the last " \sim_{\log} " equation follows from the cardinality estimate $|E_{\delta^{1/2}}| \sim \delta^{-s/2}$ and the fact that $P_{\delta^{1/2}}$ is a $(\delta^{1/2}, 1)$ -set. By discarding a constant fraction of points from $E_{\delta^{1/2}}$, one may now assume that

$$\sum_{T \in \mathcal{T}_e} \sum_{\substack{p, q \in T \cap P_{\delta^{1/2}} \\ p \neq q}} \frac{1}{|p - q|^{1-s}} \lesssim \delta^{s/2-1} \quad (4.9)$$

holds uniformly for all $e \in E_{\delta^{1/2}}$.

4.0.4. *Analysis at scale δ .* For $B \in \mathcal{B}$, recall that $P_B = P \cap B$ is a $(\delta, 1)$ -set of cardinality $|P_B| \approx \delta^{-1/2}$. I now claim the following: for fixed $B \in \mathcal{B}$, there are $\sim \delta^{-s/2}$ vectors in $E_{\delta^{1/2}}$ such that $\pi_e(P_B)$ contains a (δ, s) -set of cardinality $\approx \delta^{-s/2}$.¹ This is fairly standard, but I record the details for completeness. First, observe that $\delta^{-1/2}P_B$ is a $(\delta^{1/2}, 1)$ -set of cardinality $\approx \delta^{-1/2}$. Next, consider the measures

$$\mu_B := \frac{1}{|P_B|} \sum_{p \in \delta^{-1/2}P_B} \frac{\chi_{B(p, \delta^{1/2})}}{\delta}$$

and

$$\nu := \frac{1}{|E_{\delta^{1/2}}|} \sum_{e \in E_{\delta^{1/2}}} \frac{\chi_{B(e, \delta^{1/2}) \cap S^1}}{\delta^{1/2}},$$

and note that $\mu_B(\mathbb{R}^2) \sim 1 \sim \nu(S^1)$. For $r \geq \delta^{1/2}$, one has the uniform estimates $\mu(B(x, r)) \lesssim r$ and $\nu(B(e, r)) \lesssim r^s$, while for $0 < r \leq \delta^{1/2}$ one has the obvious improved estimates. After some straightforward computations, it follows that

$$\int_{S^1} I_s(\pi_{e\#}\mu) d\nu e := \iint \left[\int_{S^1} \frac{d\nu e}{|\pi_e(x) - \pi_e(y)|^s} \right] d\mu x d\mu y \lesssim 1. \quad (4.10)$$

Indeed, the inner integral (in brackets) can be estimated by $\lesssim \log(1/\delta)/|x - y|^s$, and then

$$\int_{S^1} I_s(\pi_{e\#}\mu) d\nu e \lesssim \log(1/\delta) \int \left[\frac{d\mu x}{|x - y|^s} \right] d\mu y \lesssim 1,$$

since the inner integral is again bounded by $\lesssim 1$ for any $y \in \mathbb{R}^2$. Consequently, $I_s(\pi_{e\#}\mu) \lesssim 1$ for a set of vectors $E_0 \subset S^1$ of ν -measure at least $1/2$. One evidently needs $\gtrsim \delta^{-s/2}$ arcs of the form $B(e, \delta^{1/2}) \cap S^1$, $e \in E_{\delta^{1/2}}$ to cover E_0 , and this gives rise to a subset $E_{\delta^{1/2}}^0 \subset E_{\delta^{1/2}}$ with $|E_{\delta^{1/2}}^0| \sim \delta^{-s/2}$. For every $e \in E_{\delta^{1/2}}^0$, there exists a vector $e' \in B(e, \delta^{1/2}) \cap S^1$ with $I_s(\pi_{e'\#}\mu) \lesssim 1$. It follows that $\mathcal{H}_\infty^s(\pi_{e'}(\text{spt } \mu)) \gtrsim 1$, hence $\pi_{e'}(\text{spt } \mu)$ contains a $(\delta^{1/2}, s)$ -set of cardinality $\approx \delta^{-s/2}$ by Proposition 2.5. Since $\pi_{e'}(\text{spt } \mu)$ is contained in the $\delta^{1/2}$ -neighbourhood of $\pi_{e'}(\delta^{-1/2}P_B)$, the same conclusion holds for $\pi_{e'}(\delta^{-1/2}P_B)$. Finally, using $|e' - e| \leq \delta^{1/2}$, the conclusion remains valid for $\pi_e(\delta^{-1/2}P_B)$, and thus $\pi_e(P_B)$ contains a (δ, s) -set of cardinality $\approx \delta^{-s/2}$ for every $e \in E_{\delta^{1/2}}^0$.

Now, let $G \subset \mathcal{B} \times E_{\delta^{1/2}}$ consist of those pairs (B, e) such that $\pi_e(P_B)$ contains a (δ, s) -set of cardinality $\approx \delta^{-s/2}$. Then, the previous argument shows that $|\{e \in E_{\delta^{1/2}} : (B, e) \in G\}| \gtrsim \delta^{-s/2}$ for every $B \in \mathcal{B}$, and consequently

$$\frac{1}{|E_{\delta^{1/2}}|} \sum_{e \in E_{\delta^{1/2}}} \sum_{B \in \mathcal{B}} \chi_G(B, e) \gtrsim |\mathcal{B}| = |P_{\delta^{1/2}}| \gtrsim_{\log} \delta^{-1/2}.$$

¹The claim is close to Proposition 2.6: the main difference is that Proposition 2.6 only requires the set of directions E to be δ -separated and of cardinality $\approx \delta^{-s}$ (as opposed to being a (δ, s) -set), but also the conclusion there does not guarantee that $\pi_e(P)$ would contain a large (δ, s) -set for any $e \in E$. In fact, easy examples show that a cardinality estimate on E alone does not yield the stronger conclusion desired here.

This implies that $|\{B \in \mathcal{B} : (B, e) \in G\}| \gtrsim_{\log} \delta^{-1/2}$ for some $e = e_0 \in E_{\delta^{1/2}}$. From this point on, the reader may forget about the rest of the vectors in $E_{\delta^{1/2}}$. Let

$$P_{\delta^{1/2}}^0 := \{p_B : (B, e_0) \in G\}, \quad (4.11)$$

so that $|P_{\delta^{1/2}}^0| \gtrsim_{\log} \delta^{-1/2}$. Recall the family of $\delta^{1/2}$ -tubes $\mathcal{T} := \mathcal{T}_{e_0}$. Let $\mathcal{T}^{\epsilon_0} := \{T \in \mathcal{T} : |T \cap P_{\delta^{1/2}}^0| \geq \delta^{2\epsilon_0 + (s-1)/2}\}$. Recalling $|\mathcal{T}| \lesssim \delta^{-s/2 - \epsilon_0}$ (by (4.7)), observe that

$$\sum_{T \in \mathcal{T} \setminus \mathcal{T}^{\epsilon_0}} |T \cap P_{\delta^{1/2}}^0| \lesssim \delta^{-s/2 - \epsilon_0} \cdot \delta^{2\epsilon_0 + (s-1)/2} = \delta^{\epsilon_0 - 1/2}.$$

Since $|P_{\delta^{1/2}}^0| \gtrsim_{\log} \delta^{-1/2}$, this implies that, for small enough $\delta > 0$, at least $|P_{\delta^{1/2}}^0|/2$ points of $P_{\delta^{1/2}}^0$ are contained in the union of the tubes in \mathcal{T}^{ϵ_0} . Replacing \mathcal{T} by \mathcal{T}^{ϵ_0} , I may – and will – henceforth assume that $|T \cap P_{\delta^{1/2}}^0| \gtrsim \delta^{(s-1)/2}$ holds uniformly for all the tubes in \mathcal{T} .

As a further regularisation, I claim that $\gtrsim |P_{\delta^{1/2}}^0|$ points in $P_{\delta^{1/2}}^0$ are contained in tubes $T \in \mathcal{T}$ satisfying the converse inequality $|T \cap P_{\delta^{1/2}}^0| \lesssim \delta^{(s-1)/2}$. This follows from (4.9). Recalling that $|T \cap P_{\delta^{1/2}}^0| \gtrsim \delta^{(s-1)/2}$ for every $T \in \mathcal{T}$, there exists $C \gtrsim \delta^{(s-1)/2}$ such that the " C -dense tubes" $T \in \mathcal{T}^C$, with $|P_{\delta^{1/2}}^0 \cap T| \sim C$, cover a total of $\gtrsim |P_{\delta^{1/2}}^0|$ points in $P_{\delta^{1/2}}^0$. Then $|\mathcal{T}^C| \approx |P_{\delta^{1/2}}^0|/C \approx \delta^{-1/2}/C$, and hence, by (4.9),

$$\delta^{s/2-1} \gtrsim \sum_{T \in \mathcal{T}^C} \sum_{\substack{p, q \in T \cap P_{\delta^{1/2}}^0 \\ p \neq q}} \frac{1}{|p - q|^{1-s}} \gtrsim \sum_{T \in \mathcal{T}^C} |P_{\delta^{1/2}}^0 \cap T|^2 \approx \frac{C^2}{\delta^{1/2}C} = \frac{C}{\delta^{1/2}}.$$

This gives $C \approx \delta^{(s-1)/2}$, as claimed. It has now been established that $\approx |P_{\delta^{1/2}}^0| \sim \delta^{-1/2}$ points of $P_{\delta^{1/2}}^0$ are covered by tubes $T \in \mathcal{T}$ satisfying

$$|P_{\delta^{1/2}}^0 \cap T| \approx \delta^{(s-1)/2}. \quad (4.12)$$

In particular, this implies that there are $\approx \delta^{-s/2}$ tubes in \mathcal{T} satisfying (4.12). Finally, using Chebyshev's inequality and (4.9), one sees that $\approx \delta^{-s/2}$ out of the tubes satisfying (4.12) also satisfy

$$\sum_{\substack{p, q \in T \cap P_{\delta^{1/2}}^0 \\ p \neq q}} \frac{1}{|p - q|^{1-s}} \lesssim \delta^{s-1}. \quad (4.13)$$

In the sequel, I am only interested in the tubes T satisfying both (4.12) and (4.13). There are $\approx \delta^{-s/2}$ such tubes, and they cover $\approx \delta^{-1/2}$ points of $P_{\delta^{1/2}}^0$. For notational convenience, I will continue denoting these tubes by \mathcal{T} .

For $T \in \mathcal{T}$, write

$$P_T := \bigcup_{p \in T \cap P_{\delta^{1/2}}^0} P_{B_p},$$

where $B_p \in \mathcal{B}$ is the unique $\delta^{1/2}$ -ball containing p (thus $p = p_{B_p}$). Let E_δ be a maximal δ -separated set inside $E \cap B(e_0, \delta^{1/2})$. Recall from Section 4.0.2 that E_δ is a (δ, s) -set with $|E_\delta| \approx \delta^{-s/2}$. For future reference, I already observe that

$$\pi_e(P_B) \subset [\pi_{e_0}(P_B)](C\delta), \quad e \in E_\delta, B \in \mathcal{B}, \quad (4.14)$$

for some absolute constant $C \geq 1$, where $A(\rho)$ stands for the ρ -neighbourhood of A . This follows from elementary geometry, recalling that $|e - e_0| \leq \delta^{1/2}$ and $\text{diam}(B) = \delta^{1/2}$ for $B \in \mathcal{B}$. Note that, for $e \in E_\delta$, the sets $\pi_e(P_T)$, $T \in \mathcal{T}$, have bounded overlap. Consequently, recalling also (4.8),

$$\delta^{-s-2\epsilon_0} \geq N(\pi_e(P), \delta) \gtrsim \sum_{T \in \mathcal{T}} N(\pi_e(P_T), \delta).$$

Hence

$$\frac{1}{|E_\delta|} \sum_{T \in \mathcal{T}} \sum_{e \in E_\delta} N(\pi_e(P_T), \delta) \lesssim \delta^{-s-2\epsilon_0}.$$

Since $|\mathcal{T}| \approx \delta^{-s/2} \sim |E_\delta|$, it follows that there is a tube $T_0 \in \mathcal{T}$ with

$$\sum_{e \in E_\delta} N(\pi_e(P_{T_0}), \delta) \lesssim \delta^{-s}. \quad (4.15)$$

By Chebyshev's inequality, there exist $\sim \delta^{-s/2}$ vectors $e \in E_\delta$ with

$$N(\pi_e(P_{T_0}), \delta) \lesssim \delta^{-s/2}. \quad (4.16)$$

Such vectors form a (δ, s) -subset of E_δ with cardinality $\sim |E_\delta|$, so one may just as well assume that every vector in E_δ satisfies (4.16).

Specify one of the vectors in E_δ , say $e_1 \in E_\delta$. For notational convenience, assume that

$$e_1 = (1, 0). \quad (4.17)$$

By the definition (4.11) of $P_{\delta^{1/2}}^0$, every projection $\pi_{e_0}(P_B)$ with $p \in T_0 \cap P_{\delta^{1/2}}^0$ contains a (δ, s) -set of cardinality $\approx \delta^{-s/2}$. Moreover, by the simple geometric observation (4.14), the same remains true for e_1 in place of e_0 . I denote by Δ_B a (δ, s) -set with $\Delta_B \subset \pi_{e_1}(P_B)$ and $|\Delta_B| \approx \delta^{-s/2}$.

Recall the inequality (4.13), and that $|P_{\delta^{1/2}}^0 \cap T_0| \approx \delta^{(s-1)/2}$ by (4.12). Using Chebyshev's inequality, one can now choose a subset $P_{\delta^{1/2}}^{T_0} \subset P_{\delta^{1/2}}^0 \cap T_0$ of cardinality $|P_{\delta^{1/2}}^{T_0}| \approx \delta^{(s-1)/2}$ such that

$$\sum_{\substack{q \in P_{\delta^{1/2}}^{T_0} \\ q \neq p}} \frac{1}{|q - p|^{1-s}} \lesssim \delta^{(s-1)/2}, \quad p \in T_{\delta^{1/2}}^{T_0}.$$

In particular, this implies that $P_{\delta^{1/2}}^{T_0}$ is a $(\delta^{1/2}, 1-s)$ -set with cardinality $\approx \delta^{(s-1)/2}$.

4.0.5. *Constructing a product-like set with small projections.* I first define a set A_1 , which will play the role of "B", once Proposition 3.1 is eventually applied. As defined in the previous section, the set $P_{\delta^{1/2}}^{T_0}$ is a $(\delta^{1/2}, 1-s)$ -set with cardinality $\approx \delta^{(s-1)/2}$. Its points are contained in the $\delta^{1/2}$ -tube T_0 perpendicular to e_0 . Since $|e_0 - e_1| \leq \delta^{1/2}$, and I assumed in (4.17) that $e_1 = (1, 0)$, this means that the projection to the y -axis restricted to $P_{\delta^{1/2}}^{T_0}$ is "nearly biLipschitz". In particular, the following holds. Write $p_B = (p_B^x, p_B^y)$. Then $\{p_B^y : p_B \in P_{\delta^{1/2}}^{T_0}\}$ contains a $(\delta^{1/2}, 1-s)$ -set A_1 of cardinality $|A_1| \approx \delta^{(s-1)/2}$.

What follows next is a construction of a "product-like" set F' with $|F'| \approx \delta^{-1/2}$ and $N(\pi_e(F'), \delta) \lesssim N(\pi_e(P_{T_0}), \delta)$ for $e \in E_\delta$. The set F' will (essentially) play the role of "P", once Proposition 3.1 is eventually applied.

The set F' is of the form

$$F' = \bigcup_{b \in A_1} A'_b \times \{b\}, \quad A'_b \subset \mathbb{R}. \quad (4.18)$$

So, I fix $b \in A_1$ and define A'_b . Note that $b = p_B^y$ for some $B \in \mathcal{B}_0$. Thus, recalling the (δ, s) -set $\Delta_B \subset \pi_{e_1}(P_B)$, I define

$$A'_b := \Delta_B. \quad (4.19)$$

Since $|A'_b| = |\Delta_B| \approx \delta^{-s/2}$, and $|A_1| \approx \delta^{(s-1)/2}$, the estimate $|F'| \approx \delta^{(s-1)/2} \delta^{-s/2} = \delta^{-1/2}$ holds, as required.

Next, it is time to control the projections of F' . Precisely, the claim is that

$$N(\pi_e(F'), \delta) \lesssim N(\pi_e(P_{T_0}), \delta) \lesssim \delta^{-s/2}, \quad e \in E_\delta. \quad (4.20)$$

Recall from (4.16) that $E_\delta \subset E \cap B(e_0, \delta^{1/2}) \subset E \cap B(e_1, 2\delta^{1/2})$ is a set of cardinality $|E_\delta| \sim \delta^{-s/2}$ such that the second inequality in (4.20) holds for all $e \in E_\delta$.

To establish the first inequality, it suffices to prove the following: for every $e \in B(e_1, 2\delta^{1/2})$ and every point $q \in F'$, there is a point $p \in P_{T_0}$ such that $|\pi_e(q) - \pi_e(p)| \lesssim \delta$. This follows easily from the construction. Every point of F' is of the form $q = (a, b)$, where $b = p_B^y$, and $a \in \Delta_B \subset \pi_{e_1}(P_B)$. Consequently, there exists a point

$$p \in P_B \subset P_{T_0}$$

such that $\pi_{e_1}(p) = a = \pi_{e_1}(q)$ and $|p - q| \lesssim \delta^{1/2}$. Moreover, it follows from $|p - q| \lesssim \delta^{1/2}$ that

$$e \mapsto \pi_e(p) - \pi_e(q) = \pi_e(p - q)$$

only varies on an interval of length $\lesssim \delta$, as e varies in $B(e_1, 2\delta^{1/2})$. This and the equation $\pi_{e_1}(p) = \pi_{e_1}(q)$ imply that $|\pi_e(p) - \pi_e(q)| \lesssim \delta$ for every $e \in B(e_1, 2\delta^{1/2})$, as required. The estimate (4.20) has been established.

4.0.6. *Dilating the product set and concluding the proof.* The main accomplishment so far has been the construction of the set F' of the form (4.18), which, by (4.20), has plenty of small projections. This almost looks like a scenario, where Theorem 3.1 can be applied. In fact, all that remains is "normalisation" in terms of a horizontal dilatation.

To this end, it is convenient to re-parametrise the projections π_e , $e \in E_\delta$, as mappings of the form $\pi_t(x, y) = x + ty$. This is entirely standard, but here are the details: given $e = (\cos \theta, \sin \theta) \in E_\delta \subset B(e_1, 2\delta^{1/2})$, note that $|\theta| \lesssim \delta^{1/2}$ by the assumption $e_1 = (1, 0)$. Hence one may assume that $\cos \theta \geq 1/2$, and

$$\pi_e(x, y) = (x, y) \cdot (\cos \theta, \sin \theta) = \frac{1}{\cos \theta} \left[x + \frac{\sin \theta}{\cos \theta} y \right] =: \frac{1}{\cos \theta} \pi_{t(e)}(x, y). \quad (4.21)$$

Now, the information that E_δ is a (δ, s) -set contained in an arc of length $\sim \delta^{1/2}$ translates to the statement that $\tilde{E}_\delta := \{t(e) : e \in E_\delta\}$ is a (δ, s) -set contained in an interval around the origin with length $\sim \delta^{1/2}$. Note that $|\tilde{E}_\delta| \sim |E_\delta| \sim \delta^{-s/2}$, and I assume for convenience assume that $\tilde{E}_\delta \subset [0, \delta^{1/2}]$. Moreover, it is obvious from (4.20) and the formula (4.21) that

$$N(\pi_t(F'), \delta) \lesssim \delta^{-s/2}, \quad t \in \tilde{E}_\delta. \quad (4.22)$$

Finally, a horizontal dilatation is applied. Consider the set

$$F := \{(\delta^{-1/2}x, y) : (x, y) \in F'\} = \bigcup_{b \in A_1} (\delta^{-1/2}A'_b) \times \{b\}.$$

To complete the proof, observe that each set $A_b := \delta^{-1/2}A'_b$, $b \in A_1$, is a $(\delta^{1/2}, s)$ -set of cardinality $\approx \delta^{-s/2}$. The cardinality claim is clear from the definition (4.19), while the $(\delta^{1/2}, s)$ -set property follows from

$$|B(x, r) \cap A_b| = |B(\delta^{1/2}x, \delta^{1/2}r) \cap A'_b| \lesssim \left(\frac{\delta^{1/2}r}{\delta} \right)^s = \left(\frac{r}{\delta^{1/2}} \right)^s, \quad r \geq \delta^{1/2}. \quad (4.23)$$

Consequently, the sets A_1 (as " B "), A_b , $b \in A_1$, and F (as " P ") satisfy the hypotheses of Proposition 3.1 at scale $\delta^{1/2}$. Moreover, F has plenty of small projections. Consider the set $\tilde{E}_{\delta^{1/2}} := \delta^{-1/2}\tilde{E}_\delta \subset [0, 1]$. Given $t' = \delta^{-1/2}t \in \tilde{E}_{\delta^{1/2}}$ and $(\delta^{-1/2}x, y) \in F$, observe that

$$\pi_{t'}(\delta^{-1/2}x, y) = \delta^{-1/2}x + \delta^{-1/2}ty = \delta^{-1/2}\pi_t(x, y).$$

Recalling (4.22), it follows immediately that

$$N(\pi_{t'}(F), \delta^{1/2}) = N(\pi_t(F'), \delta) \lesssim \delta^{-s/2}, \quad t' \in \tilde{E}_{\delta^{1/2}}. \quad (4.24)$$

The set $\tilde{E}_{\delta^{1/2}}$ is clearly (or see (4.23)) a $(\delta^{1/2}, s)$ -set with $|\tilde{E}_{\delta^{1/2}}| \sim \delta^{-s/2}$. Consequently, (4.24) should not be possible by Proposition 3.1. A contradiction is thus reached, and the proof of Theorem 2.1 is complete.

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